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Research Paper

Distortion risk measures for nonnegative multivariate risks

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ABSTRACT

We apply distortion functions to bivariate survival functions for nonnegative random variables. This leads to a natural extension of univariate distortion risk measures to the multivariate setting. For Gini's principle, the proportional hazard transform distortion and the dual power transform distortion, certain families of multivariate distributions lead to a straightforward risk measure. We show that an exact analytical expression can be obtained in some cases. We consider the independence case, the bivariate Pareto distribution and the bivariate exponential distribution. An illustration of the estimation procedure and the interpretation is also included. In the case study, we consider two loss events with a single risk value and monitor the two events together over four different periods. We conclude that the dual power transform gives more weight to the observations of extreme losses, but that the distortion parameter can modulate this influence in all cases. In our example, multivariate risk clearly diminishes over time.

Keywords: distortion functions; multivariate risk; multiperiod risk assessment; dependence; risk aggregation; multivariate loss.

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1 INTRODUCTION

Classical risk measures are defined on univariate risks, ie, on unidimensional random variables, and not on a multivariate setting. However, risk evaluation problems in real life are rarely one dimensional. In many practical applications, it is usual to deal with multidimensionality by transforming multivariate risks into a unidimensional risk using some aggregation procedure, for instance using the sum of risks. Once the multiple dimensions of the risk problem have been reduced to one dimension, classical risk measures can be used to quantify the risk.

This paper takes a different perspective in proposing a set of risk measures for nonnegative multivariate risks. Our approach to multivariate risk assessment problems differs from the traditional procedure in the way aggregation is performed: instead of transforming the multivariate random variable first and then quantifying the risk in the univariate setting, we concentrate on the whole multidimensional distribution and define a one-dimensional risk measurement value for the distribution. We follow the definition given by Rüschendorf (2013, p. 180), which we present in Section 2.

Risk management often requires multivariate risk measures that capture the interdependence between many risk factors. When considering all the dimensions, it is natural to take the joint multivariate distribution function of the risks as the starting point. For instance, the quantile of the joint distribution leads to the analysis of critical layers (as defined by Salvadori *et al* (2011) and discussed later by Di Bernardino and Palacios-Rodríguez (2017)), which are multidimensional by definition. Our approach is totally different: we aim to obtain a single value that summarizes the risk of a multivariate random vector, but we apply a distortion to the joint survival multidimensional function and then we carry out a multiple integration in order to obtain a summary value. The main advantage is that we do not work with vectors of risk measures. Moreover, we show that, for some special multivariate distributions, this approach provides simple analytical expressions. A potential drawback is that the distortion of the multivariate survival and the multiple integral, even if it is an elegant generalization, is a summary measure that combines all dimensions in one and may be difficult to interpret.

As stated in Embrechts and Puccetti (2006), in the risk management and finance literature random vectors are referred to as portfolios, and individual random subvectors as risks. Usually portfolios of identically distributed, nonnegative risks are considered. Note that even if financial returns can be positive or negative, the risk manager looks at losses, so that one of the two axes is of particular interest. According to Sun *et al* (2017), portfolio risk management measures the distribution of losses in a portfolio over a fixed horizon, but the dependence between risk factors complicates the computation. The dependence structure is then assumed from a joint multivariate distribution that has a fixed dependence over time, or a multivariate copula function that could include some time varying dependence. Alternatively, in order to analyze each dimension separately, we must take the marginal distribution or the componentwise measures. Cousin and Di Bernardino (2013) dealt with multidimensionality by analyzing vector-valued measures with the same dimension as the underlying risk variables; this approach is also referred to as set-valued risk measures. From the vector of risk measures, Cousin and Di Bernardino define the lower-orthant value-at-risk (VaR), which is constructed from level sets of multivariate distribution functions, and the upper-orthant VaR, which is constructed from level sets of multivariate survival functions.

We should note that an application of multivariate risk measures is found in the risk management of financial institutions, since Basel III requires a minimum capital that is derived from the analysis of risk on an aggregated basis. Traditional univariate risk measures cannot address portfolio risk management as a whole.

The set of risk measures we propose can be called distortion risk measures for nonnegative multivariate risks. As explained in the following sections, there is a natural parallelism between the unidimensional distortion risk measures introduced by Wang (1995a,b) and the risk measures introduced in this paper.

In the insurance setting, and in operational risk in particular, risk managers generally look at losses only, and these are positive values. If these results were to be extended to the analysis of returns, which can be either positive or negative, then the same principle of distortion as for the joint survival could be used. Belles-Sampera *et al* (2013) indicated that distortion risk measures can be interpreted as aggregation operators for finite random variables that do not necessarily have to be positive.

We show in the illustrations that our proposal provides a good method to monitor multivariate risks that can be especially interesting in the context of operational risk analysis.

2 DISTORTION RISK MEASURES FOR THE NONNEGATIVE UNIVARIATE CASE

Let us assume a probability space $(\Omega, \mathcal{A}, P)$ with sample space Ω, a σ-algebra $\mathcal{A}$ and a probability P from $\mathcal{A}$ to [0, 1], and the set of all random variables defined on this space. Consider a nonnegative random variable X defined on this probability space and its survival function $S(x) = P(X > x)$. A distortion risk measure applied to X, which we denote by $\psi[g:S]$, is defined by

$$\psi[g:S] = \int_0^{+\infty} g(S(x))\,\mathrm{d}x, \tag{2.1}$$

where g is the associated distortion function, which is a function from [0, 1] to [0, 1], and it must be increasing (not necessarily strictly increasing) and such that $g(0) = 0$

TABLE 1 Some examples of distortion functions for distortion risk measures.

Risk measure	Parameters	Distortion function
Gini's principle	$0 < \theta < 1$	$g_\theta(t) = (1+\theta)t - \theta t^2$
Proportional hazard transform	$m \geqslant 1$	$g_m(t) = t^{1/m}$
Dual power transform	$m \geqslant 1$	$g_m(t) = 1 - (1-t)^m$

and $g(1) = 1$. Two main examples of distortion risk measures broadly used in financial and insurance applications are VaR and tail VaR (TVaR) at a fixed confidence level $\alpha \in (0, 1)$, whose distortion functions are

$$\delta_\alpha(t) = \mathbf{1}_{[1-\alpha,1]}(t) \quad \text{and} \quad \gamma_\alpha(t) = \frac{t}{1-\alpha}\mathbf{1}_{[0,1-\alpha)}(t) + \mathbf{1}_{[1-\alpha,1]}(t),$$

respectively, where $\mathbf{1}_{[a,b]}(t)$ equals 1 if $a \leqslant t \leqslant b$, and 0 otherwise. Three classes of distortion risk measures that will be used in the rest of the paper and their associated distortion functions are given in Table 1, namely the risk measure based on Gini's principle, the proportional hazard transform and the dual power transform. Similar procedures can be applied to Denneberg's absolute deviation principle (Denneberg 1990), which is defined through the distortion function $g_\alpha(t) = t(1+\alpha)\mathbf{1}_{[0,0.5)}(t) + (\alpha + (1-\alpha)t)\mathbf{1}_{[0.5,1]}(t)$, and to the GlueVaR risk measures introduced by Belles-Sampera *et al* (2014), which generalize range VaR and follow from the distortion functions

$$g_{\beta,\alpha}^{h_1,h_2}(t) = \left(\frac{h_1}{(1-\beta)}\right)\mathbf{1}_{[0,1-\beta)}(t) + \left(h_1 + \frac{(h_2-h_1)}{(\beta-\alpha)}[t-(1-\beta)]\right)\mathbf{1}_{[1-\beta,1-\alpha)}(t) + \mathbf{1}_{[1-\alpha,1]}(t),$$

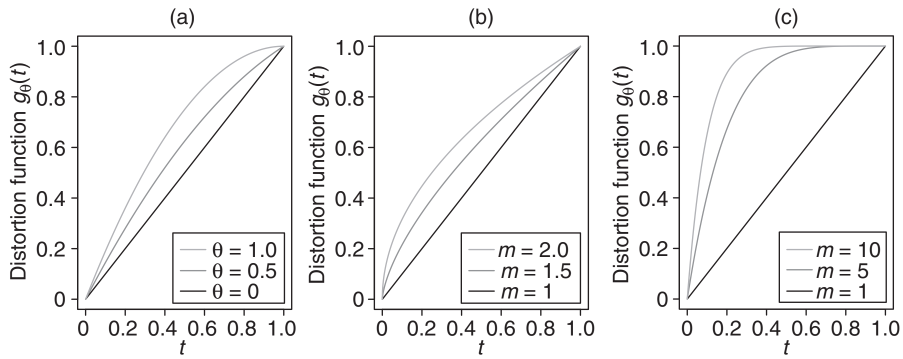
with $\alpha \leqslant \beta < 1$, $0 < h_1 < 1$, $h_1 \leqslant h_2 < 1$. Note that when $\alpha = \beta$ neither $(\beta-\alpha)^{-1}$ nor $\mathbf{1}_{[1-\beta,1-\alpha)}(t)$ for $t \in (0, 1)$ are well defined and, in addition, $\alpha = \beta$ implies that $h_1 = h_2$ because h_1 and h_2 represent the distorted survival probability associated with $1-\beta$ and $1-\alpha$, respectively. So, in those cases, the distortion function reduces to

$$g_\alpha^{h_1}(t) = \left(\frac{h_1}{(1-\alpha)}\right)\mathbf{1}_{[0,1-\alpha)}(t) + \mathbf{1}_{[1-\alpha,1]}(t).$$

Expression (2.1) can be understood as the Choquet integral of X with respect to the set function $g \circ P$, where P is the probability function associated with the probability space in which X is defined (Choquet 1954; Denneberg 1994). Henceforth, only nonnegative random variables are considered.

The specific preference for a distortion function is difficult to determine. However, the transformation of the survival function reflects in some way the emphasis on the

FIGURE 1 Distortion functions corresponding to (a) Gini's principle, (b) proportional hazard transform and (c) dual power transform.



extremes. Belles-Sampera *et al* (2016) examined how risk attitudes can be represented in the selection of a given distortion. They showed that the analysis of the distortion function offers a local description of the agent's stance on risk in relation to the occurrence of accumulated losses. Here, the concepts of absolute risk attitude and local risk attitude arise naturally. For example, the area under the distortion reveals the global risk attitude, whereas the ratio of the distortion to the identity function provides us with local information about the relative risk behavior associated with the risk measure at any point in the range of values.

A plot of the three distortion functions (see Figure 1) shows that the Gini principle risk measure weights the right tail less heavily than the other measures because its distortion function is flatter than the others for low values. When the proportional hazard transform is used, the importance of the large losses is moderate, but when the dual power transform is selected with parameter equal to 5 or 10 we observe a high curve for low values, which means that the right tail of the positive losses will have more importance for the calculation of the risk measure. Therefore, extreme losses are weighted more than in other cases for the dual power transform for $m = 10$, because the distortion function is closer to 1 for low values of t.

3 DISTORTION RISK MEASURES FOR THE NONNEGATIVE BIVARIATE CASE

Let $(X_1, X_2)^{\mathrm{T}}$ be a nonnegative bivariate random variable with joint survival function $S_{12}(x_1, x_2)$ and marginal survival functions $S_1(x_1)$ and $S_2(x_2)$.

The idea is to introduce distortion risk measures defined on $(X_1, X_2)^{\mathrm{T}}$ that are congruent with the unidimensional distortion risk measures defined on the associated marginal distributions.

In a first step, we consider a distortion function $g(\cdot)$, and we define a distorted bivariate survival as follows:

$$T_{12}(x_1, x_2) = g[S_{12}(x_1, x_2)], \tag{3.1}$$

where the $g(\cdot)$ function is chosen in order to define a genuine bivariate survival function in (3.1). Note that the marginal survival functions in (3.1), corresponding to distorted transformations of the joint survival function $S_{12}(x_1, x_2)$, are

$$T_1(x_1) = T_{12}(x_1, 0) = g[S_1(x_1)] \tag{3.2}$$

and

$$T_2(x_2) = T_{12}(0, x_2) = g[S_2(x_2)]. \tag{3.3}$$

Once a suitable distortion function $g(\cdot)$ has been selected, a possible distortion risk measure associated with (3.1) is simply

$$\psi_{12}[g{:}S_{12}] = \int_0^{\infty}\int_0^{\infty} T_{12}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_0^{\infty}\int_0^{\infty} g[S_{12}(x_1, x_2)]\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{3.4}$$

Note that the corresponding distortion risk measures associated with (3.2) and (3.3) are

$$\psi_1[g{:}S_1] = \int_0^{\infty} g[S_1(x_1)]\,\mathrm{d}x_1 \tag{3.5}$$

and

$$\psi_2[g{:}S_2] = \int_0^{\infty} g[S_2(x_2)]\,\mathrm{d}x_2. \tag{3.6}$$

So, there is a natural parallelism between the multivariate setting (3.4) and the marginals in (3.5) and (3.6) and the univariate case. This approach is the one proposed in Rüschendorf (2006, Section 3) and Rüschendorf (2013, p. 180). However, it is not the only possible way to address risk measures for bivariate risks (see, for instance, Embrechts *et al* (2009), which shows how multivariate extreme value theory yields the ideal modeling environment). Different extensions to multivariate risk measurement using VaR and TVaR can be found in Cousin and Di Bernardino (2013, 2014), where vector-valued measures are proposed with the same dimension as the underlying risk portfolio, and the lower-orthant (upper-orthant) risk measure is constructed from level sets of multivariate distribution functions (multivariate survival distribution functions). Unlike allocation measures or systemic risk measures, these measures are suitable for multivariate risk problems where risks are heterogeneous and cannot be aggregated together before calculating the risk measure.

4 SOME BIVARIATE DISTORTION RISK MEASURES WITH A CLOSED-FORM EXPRESSION

Before generalizing this definition to higher dimensions, we explore some expressions for $\psi_{12}[g{:}S_{12}]$ where the g function has been restricted to belong to the set of distortion functions associated with univariate risk measures presented in Table 1. Some of the cases considered here have the advantage of providing a straightforward analytical expression. The main reason why having closed-form expressions is interesting is because these risk measures can then be implemented in spreadsheet calculations and simulation procedures very easily.

Let us begin with a bivariate random variable $(X_1, X_2)^{\mathrm{T}}$ with independent marginals, and then assume a dependence structure between the marginals driven by copulas in the Farlie–Gumbel–Morgenstern (FGM) family. In the case of independence, we do not assume any particular marginal distribution, but this situation is not the main focus of the paper, because what we really want to analyze is cases where we assume a dependence structure. The bivariate Pareto distribution is a clear example of the type of two-dimensional distribution that a risk manager would use to analyze losses coming from two lines of business, or two types of risk. For example, in operational risk, we can assume that losses can be of two types and therefore each severity is represented by one of the two dimensions. Similarly, the bivariate exponential distribution or the FGM distribution could reflect the monthly size of losses in, for instance, internal and external fraud.

A bivariate Pareto distribution is a standard choice for finance/insurance losses. For instance, Embrechts and Puccetti (2006) calculate the bounds of a sum of two Pareto and lognormal bivariate risks, and provide a new definition of multivariate VaR.

4.1 Risk measures for the bivariate case assuming independence

Let $(X_1, X_2)^{\mathrm{T}}$ be a bivariate risk with joint survival function $S_{12}(x_1, x_2)$. In this section, we obtain bivariate risk measures assuming independence between marginal risks X_1 and X_2, that is, assuming that $S_{12}(x_1, x_2) = S_1(x_1)S_2(x_2)$. We consider three different distortion risk measures.

4.1.1 Risk measures based on Gini's principle

Let us consider the distortion function given by Gini's principle, $g_\theta(t) = (1+\theta)t - \theta t^2$, with $0 < \theta < 1$. Using (3.4), we obtain the multivariate measure

$$\psi_{12}[g_\theta{:}S_{12}] = (1+\theta)\mu_1\mu_2 - \theta\mu_{1:2}^{(1)}\mu_{1:2}^{(2)}, \tag{4.1}$$

where $\mu_i = E(X_i)$, $i = 1, 2$, and $\mu_{1:2}^{(i)}$, $i = 1, 2$, represent the mathematical expectations of the minimum of two copies of the random variable X_i, with $i = 1, 2$.

4.1.2 Risk measures based on the proportional hazard transform

Let us consider the proportional hazard transform principle given by the distortion function $g_m(t) = t^{1/m}$, $m \geqslant 1$. In this case, using the notation $F_i(x_i) = 1 - S_i(x_i)$, $i = 1, 2$, the multivariate risk measure can be written as

$$\psi_{12}[g_m{:}S_{12}] = \prod_{i=1}^{2} E\left\{ F_i^{-1}\left[\mathrm{Be}\left(1, \frac{1}{m}\right)\right]\right\}, \tag{4.2}$$

where $\mathrm{Be}(a, b)$ represents a classical beta distribution. Note that the terms in the product correspond to the mathematical expectation of the generalized beta distribution (see Alexander *et al* 2012; Jones *et al* 2004) with baseline cumulative distribution function F_i and parameters $(1, 1/m)$.

4.1.3 Risk measures based on the dual power transform

The following bivariate risk measure is based on the dual power transform principle: $g_m(t) = 1 - (1 - t)^m$ with $m \geqslant 1$. The corresponding multivariate risk measure is given by

$$\psi_{12}[g_m{:}S_{12}] = \sum_{k=1}^{m} (-1)^{k+1} \binom{m}{k} \mu_{1:k}^{(1)} \mu_{1:k}^{(2)}, \tag{4.3}$$

where $\mu_{1:k}^{(i)}$, $i = 1, 2$, represent the mathematical expectations of the minimum of k independent and identically distributed (iid) copies of the random variable X_i, with $i = 1, 2$. Note that $\mu_{1:1}^{i} = \mu_i$ for all i.

4.2 Risk measures for the bivariate Pareto distribution

The examples of bivariate risk measures with a closed-form expression that are presented in Section 4.1 are based on the hypothesis of the independence of both risks. However, this assumption is often unrealistic in practice because losses from different sources may occur simultaneously. Then, we work with different classes of dependent risks.

In this section we consider the expressions that several bivariate distortion risk measures take when they are applied to a bivariate dependent Pareto distribution as proposed by Mardia (1962) (see also Arnold 1983), which is sometimes also called the bivariate Lomax distribution. The bivariate Pareto distribution is defined in terms of the following bivariate survival function:

$$S_{12}(x_1, x_2) = \left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2}\right)^{-a}, \quad x_1, x_2 \geqslant 0, \tag{4.4}$$

where $\sigma_1, \sigma_2 > 0$ are scale parameters and $a > 0$ is a shape parameter. Note that both marginal distributions are Pareto distributions with survival functions equal to $S_i(x_i) = 1/(1 + x_i/\sigma_i)^a$, with $x_i \geqslant 0$, $i = 1, 2$.

To compute the different bivariate risk measures, we use the result of Lemma 4.1.

LEMMA 4.1 *If $\sigma_1, \sigma_2 > 0$ and $a > 2$, then*

$$\int_0^\infty \int_0^\infty \mathrm{d}x_1\,\mathrm{d}x_2 \left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2}\right)^{-a} = \frac{\sigma_1\sigma_2}{(a-1)(a-2)}. \tag{4.5}$$

PROOF The result is direct, taking into account that if $a > 1$, then

$$\int_0^\infty \mathrm{d}x_1 \left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2}\right)^{-a} = \frac{\sigma_1}{a-1}\left(1 + \frac{x_2}{\sigma_2}\right)^{-(a-1)}.$$

□

4.2.1 Risk measures based on Gini's principle

If we consider the distortion function given by Gini's principle, $g_\theta(t) = (1+\theta)t - \theta t^2$, with $0 < \theta < 1$, using (3.4) and (4.5) we obtain

$$\psi_{12}[g_\theta{:}S_{12}] = (1+\theta)\frac{\sigma_1\sigma_2}{(a-1)(a-2)} - \theta\frac{\sigma_1\sigma_2}{(2a-1)(2a-2)},$$

which can be written as

$$\psi_{12}[g_\theta{:}S_{12}] = \frac{(3a\theta + 4a - 2)\sigma_1\sigma_2}{2(a-1)(a-2)(2a-1)} \tag{4.6}$$

and is valid for $a > 2$.

4.2.2 Risk measures based on the proportional hazard transform

Now, we choose the proportional hazard transform principle represented by the distortion function $g_m(t) = t^{1/m}$ with $m \geqslant 1$. The associated risk measure is

$$\psi_{12}[g_m{:}S_{12}] = \frac{m\sigma_1\sigma_2}{(a-m)(a-2m)} \tag{4.7}$$

if $a > 2m$.

4.2.3 Risk measures based on the dual power transform

For the dual power transform principle with distortion function $g_m(t) = 1 - (1-t)^m$ with $m \geqslant 1$, the corresponding bivariate risk measure is given by

$$\psi_{12}[g_m{:}S_{12}] = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\frac{\sigma_1\sigma_2}{(ak-1)(ak-2)} \tag{4.8}$$

with $a > 2$.

4.3 Risk measures for the bivariate exponential distribution

Another dependence structure to be investigated is the bivariate exponential distribution given by

$$S_{12}(x_1, x_2) = \exp(-a_1x_1 - a_2x_2 - \phi a_1a_2x_1x_2), \quad x_1, x_2 \geqslant 0, \tag{4.9}$$

where $a_1, a_2 > 0$ and $0 \leqslant \phi \leqslant 1$. This joint survival function corresponds to the Gumbel type-I bivariate exponential distribution considered by Gumbel (1960).

The following lemma is useful for the computation of the different risk measures when they are applied to this distribution.

LEMMA 4.2 *If $S_{12}(x_1, x_2)$ denotes the bivariate survival function defined in (4.9), we have*

$$\int_0^\infty \int_0^\infty S_{12}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \frac{1}{\phi a_1 a_2}\left(\exp\left(\frac{1}{\phi}\right)\right)\left[-E\mathrm{i}\left(\frac{1}{\phi}\right)\right], \tag{4.10}$$

where

$$-E\mathrm{i}(z) = \int_z^\infty \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \tag{4.11}$$

represents the exponential integral function.

PROOF Integrating (4.9) with respect to x_1, we have

$$\int_0^\infty S_{12}(x_1, x_2)\,\mathrm{d}x_1 = \frac{\mathrm{e}^{-a_2x_2}}{a_1(1 + a_2\phi x_2)},$$

and integrating again with respect to x_2 we obtain (4.10), using definition (4.11). □

4.3.1 Risk measures based on Gini's principle

For Gini's principle, we have that the risk measure expression for the bivariate exponential distribution is

$$\begin{aligned}\psi_{12}[g_\theta{:}S_{12}] = \frac{1}{\phi a_1 a_2}\left((1+\theta)\exp\left(\frac{1}{\phi}\right)\right)\left[-E\mathrm{i}\left(\frac{1}{\phi}\right)\right] \\ - \frac{1}{2\phi a_1 a_2}\left(\theta\exp\left(\frac{2}{\phi}\right)\right)\left[-E\mathrm{i}\left(\frac{2}{\phi}\right)\right].\end{aligned} \tag{4.12}$$

4.3.2 Risk measures based on the proportional hazard transform

In the case of the proportional hazard transform we obtain that the bivariate risk measure can be expressed as

$$\psi_{12}[g_m{:}S_{12}] = \frac{1}{\phi a_1 a_2}\left(m\exp\left(\frac{1}{m\phi}\right)\right)\left[-E\mathrm{i}\left(\frac{1}{m\phi}\right)\right]. \tag{4.13}$$

4.3.3 Risk measures based on the dual power transform principle

For the dual power transform principle we obtain the following closed-form expression for the risk measure applied to a bivariate exponential distribution:

$$\psi_{12}[g_m{:}S_{12}] = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\frac{1}{k\phi a_1 a_2}\left(\exp\left(\frac{k}{\phi}\right)\right)\left[-E\mathrm{i}\left(\frac{k}{\phi}\right)\right]. \tag{4.14}$$

4.4 A dependent model based on the FGM distributions

Now, we consider the Farlie–Gumbel–Morgenstern distribution (Farlie 1960; Gumbel 1960; Morgenstern 1956) with joint survival function

$$S_{12}(x_1, x_2; \alpha) = S_1(x_1)S_2(x_2)[1 + \delta(1 - S_1(x_1))(1 - S_2(x_2))], \tag{4.15}$$

where $\delta \in [-1, 1]$ is the dependence parameter, and $\delta = 0$ corresponds to the independent case.

To obtain the different bivariate risk measures, we need the following lemma.

LEMMA 4.3 *Let $X_{i:n}$ be the ith order statistics in a sample of size n, and let the ith spacing be*

$$S_{i:n} = X_{i+1:n} - X_{i:n}. \tag{4.16}$$

The fundamental formulas for moments of order statistics in terms of integrals concerning distribution function only are given by

$$E(S_{i:n}) = \binom{n}{i}\int_{-\infty}^{\infty} F(x)^i[1 - F(x)]^{n-i}\,\mathrm{d}x. \tag{4.17}$$

PROOF See Pearson (1902), and also Jones and Balakrishnan (2002). □

Using Lemma 4.3, if X is a positive random variable, we have

$$\int_0^{\infty} F(x)[1 - F(x)]\,\mathrm{d}x = \frac{E(S_{1:2})}{2} = \frac{E(X_{2:2} - X_{1:2})}{2}, \tag{4.18}$$

$$\int_0^{\infty} F(x)[1 - F(x)]^2\,\mathrm{d}x = \frac{E(S_{1:3})}{3} = \frac{E(X_{2:3} - X_{1:3})}{3}, \tag{4.19}$$

$$\int_0^{\infty} F(x)^2[1 - F(x)]^2\,\mathrm{d}x = \frac{E(S_{2:4})}{6} = \frac{E(X_{3:4} - X_{2:4})}{6}. \tag{4.20}$$

4.4.1 Risk measures based on Gini's principle

We consider the distortion function based on Gini's principle, given by $g_\theta(t) = (1 + \theta)t - \theta t^2$. We have the following theorem.

THEOREM 4.4 *Let $(X_1, X_2)^{\mathrm{T}}$ be a bivariate random variable with bivariate survival function given by (4.15). Then, we have*

$$\int_0^\infty \int_0^\infty S_{12}(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 = \mu_1\mu_2 + \tfrac{1}{4}\delta E(S_{1:2}^{(1)})E(S_{1:2}^{(2)}) \tag{4.21}$$

and

$$\begin{aligned}\int_0^\infty \int_0^\infty & S_{12}(x_1, x_2)^2\mathrm{d}x_1\,\mathrm{d}x_2 \\ &= \mu_{1:2}^{(1)}\mu_{1:2}^{(2)} + \tfrac{2}{9}\delta E(S_{1:3}^{(1)})E(S_{1:3}^{(2)}) + \tfrac{1}{36}\delta^2 E(S_{2:4}^{(1)})E(S_{2:4}^{(2)}),\end{aligned} \tag{4.22}$$

where $S_{i:n}^{(k)}$, $k = 1, 2$, is defined in (4.16), the superscript corresponds to the marginal X_k, $k = 1, 2$, and $\mu_{1:2}^{i}$, $i = 1, 2$, are defined as in Section 4.1.1.

PROOF The proof is direct using the expression for the survival FGM copula, ie, (4.15), and (4.18)–(4.20). □

Using the above result, the corresponding bivariate risk measure is

$$\begin{aligned}\psi_{12}[g_\theta{:}S_{12}] = (1 + \theta)\{\mu_1\mu_2 &+ \tfrac{1}{4}\delta E(S_{1:2}^{(1)})E(S_{1:2}^{(2)})\} \\ &- \theta\{\mu_{1:2}^{(1)}\mu_{1:2}^{(2)} + \tfrac{2}{9}\delta E(S_{1:3}^{(1)})E(S_{1:3}^{(2)}) + \tfrac{1}{36}\delta^2 E(S_{2:4}^{(1)})E(S_{2:4}^{(2)})\}.\end{aligned} \tag{4.23}$$

4.4.2 Risk measures based on the dual power transform

If we take the distortion function $g_m(t) = 1 - (1 - t)^m$ with $m = 2$, we obtain

$$\begin{aligned}\psi_{12}[g_2{:}S_{12} : \delta] = 2\mu_1\mu_2 &- \mu_{1:2}^{(1)}\mu_{1:2}^{(2)} \\ &+ \tfrac{2}{4}\delta(\mu_{2:2}^{(1)} - \mu_{1:2}^{(1)})(\mu_{2:2}^{(2)} - \mu_{1:2}^{(2)}) \\ &- \tfrac{2}{9}\delta(\mu_{2:3}^{(1)} - \mu_{1:3}^{(1)})(\mu_{2:3}^{(2)} - \mu_{1:3}^{(2)}) \\ &- \tfrac{1}{36}\delta^2(\mu_{3:4}^{(1)} - \mu_{2:4}^{(1)})(\mu_{3:4}^{(2)} - \mu_{2:4}^{(2)}),\end{aligned} \tag{4.24}$$

where

$$\mu_{i:j}^{(k)} = E[X_{i:j}^{(k)}], \quad k = 1, 2,$$

and $X_{i:j}^{(k)}$, $k = 1, 2$, denotes the ith order statistics in a sample of size j corresponding to the random variables X_1 and X_2.

If we set $\delta = 0$ in (4.24), we obtain (4.3), taking into account that $\mu_{1:1}(i) = \mu_i$ for all $i = 1, 2$.

5 EXTENSION TO THE MULTIVARIATE CASE

Let us consider a p-dimensional nonnegative random variable $(X_1, X_2, \dots, X_p)^{\mathrm{T}}$ and a distortion function g. Analogously to the definition of distortion risk measures for the nonnegative bivariate case given in (3.4), the distortion risk measure for multivariate risks associated with g may be defined as follows.

DEFINITION 5.1 A distortion risk measure for multivariate nonnegative risks can be defined as

$$\psi_{12\cdots p}[g{:}S_{12\cdots p}] = \int_0^\infty \cdots \int_0^\infty g[S_{12\cdots p}(x_1, \dots, x_p)]\,\mathrm{d}x_1 \cdots \mathrm{d}x_p, \tag{5.1}$$

where $S_{12\cdots p}$ is the multivariate survival function of the p-dimensional nonnegative random variable $(X_1, X_2, \dots, X_p)^{\mathrm{T}}$, and g is a distortion function.

This definition corresponds to that given in Rüschendorf (2006, Section 3).

Definition 5.1 may not be particularly appropriate for some purposes. For instance, if an insurance company needs to determine solvency capital for a three-year window, it is necessary that the risk value preserves the scale, so it should correspond to monetary units and not, for instance, to “monetary units to the power of three”. If X_s is the random loss from period $s-1$ to period s, $s = 1, 2, 3$, then an insurance company interested in a risk measure for vector $(X_1, X_2, X_3)^{\mathrm{T}}$ may find that $\psi_{123}[g{:}S_{123}]$ is too large. Our proposal is to consider $(\psi_{123}[g{:}S_{123}])^{1/3}$ to overcome such an inconvenience.

DEFINITION 5.2 A rescaled distortion risk measure for multivariate nonnegative risks can be defined as

$$\rho_{12\cdots p}[g{:}S_{12\cdots p}] = (\psi_{12\cdots p}[g{:}S_{12\cdots p}])^{1/p}, \tag{5.2}$$

where $\psi_{12\cdots p}[g{:}S_{12\cdots p}]$ comes from Definition 5.1, $S_{12\cdots p}$ is the multivariate survival function of the p-dimensional nonnegative random variable $(X_1, X_2, \dots, X_p)^{\mathrm{T}}$ and g is a distortion function.

Note that, once a distortion function g has been selected, Definitions 5.1 and 5.2 are both consistent with the definition of a distortion risk measure for the unidimensional case, because $\rho_1[g{:}S_1] = \psi_1[g{:}S_1]$ by (5.2), and they also match (2.1).

Standardized data could be used when the different units of measurement are a concern. In many cases the dimensions use different units of measurement. For instance, in the financial services industry, some risks are price based (such as the betas), whereas others are calculated as an index (composite indicator of systemic stress) or are balance-sheet based (the ratio of nonperforming loans to total loans).

In this section we compute only the multivariate risk measure (5.1), assuming that the components of the random vector $(X_1, \dots, X_p)^{\mathrm{T}}$ are independent, because in this case the expressions are straightforward.

For the distortion function given by Gini's principle, $g_\theta(t) = (1+\theta)t - \theta t^2$, $0 < \theta < 1$, (5.1) turns into

$$\psi_{12\cdots p}[g_\theta{:}S_{12\cdots p}] = (1+\theta)\prod_{i=1}^{p}\mu_i - \theta\prod_{i=1}^{p}\mu_{1:2}^{(i)}, \tag{5.3}$$

where $\mu_i = E(X_i)$, $i = 1, 2$, and $\mu_{1:2}^{(i)}$, $i = 1, 2, \dots, p$, represent the mathematical expectations of the minimum of two copies of the random variable X_i, with $i = 1, 2$.

In this case, the closed-form expression of the multivariate risk measure for independent risks is

$$\psi_{12\cdots p}[g_m{:}S_{12\cdots p}] = \prod_{i=1}^{p} E\left\{F_i^{-1}\left[\mathrm{Be}\left(1, \frac{1}{m}\right)\right]\right\}, \tag{5.4}$$

where $\mathrm{Be}(a, b)$ represents a classical beta distribution.

For the dual power transform principle $g_m(t) = 1 - (1-t)^m$ with $m \geqslant 1$, the expression for the multivariate risk measure is given by

$$\psi_{12\cdots p}[g_m{:}S_{12\cdots p}] = \sum_{k=1}^{m}(-1)^{k+1}\binom{m}{k}\prod_{i=1}^{p}\mu_{1:k}^{(i)}, \tag{5.5}$$

where $\mu_{1:k}^{(i)}$, $i = 1, 2, \dots, p$, represent the mathematical expectation of the minimum of k (iid) copies of the random variable X_i, with $i = 1, 2, \dots, p$.

6 A NUMERICAL EXAMPLE FOR BIVARIATE NONNEGATIVE RISKS

We considered an example where the occurrence of two phenomena is observed over time. Our objective was to provide a multivariate risk measure in order to monitor the evolution of risk of these two magnitudes using a single synthesized value. Therefore, one multivariate risk measure is better than using two different risk measures for each dimension separately.

This application shows that it is possible to analyze multivariate operational risk from many sources, for instance, when the risk manager has to monitor the occurrence of operational events by looking at the number or severity of events by class, ie, several dimensions, and wants to have only one risk value instead of a different risk measure for every type of event.

6.1 Data and methodology

For illustrative purposes, we obtained accidental (unintentional injury) death data from the Spanish national statistics institute (Instituto Nacional de Estadistica; www.ine.es). In this data set, causes of accidental death in Spain are classified as follows:

(1) traffic accidents of motor vehicles;

(2) other transport accidents;

(3) accidental falls;

(4) accidental drowning, immersion or suffocation;

(5) accidents by fire, smoke or hot substances;

(6) accidental poisoning by psychoactive drugs or abuse of drugs;

(7) other accidental poisoning;

(8) other accidents.

For this work, we grouped these into two classes: deaths due to crashes (causes (1)–(2)) and deaths due to other accidental causes (causes (3)–(8)). Then, we analyzed the following bivariate variable: the number of fatalities due to crashes (X_1) and the number of deaths due to other accidental causes (X_2) in a province or autonomous city (according to the province of residence of the deceased) per year; there are fifty provinces and two autonomous cities in Spain. For this, we selected the years 2000, 2004, 2008 and 2012. Table 2 shows the data set considered, and Figure 2 shows the corresponding three-dimensional histograms.

Given that the observed number of occurrences is large, we did not fit a discrete distribution, but fitted the bivariate Pareto distribution described in Section 4.1 by maximum likelihood. For this model, the probability density function is

$$f_{12}(x_1, x_2; a, \sigma_1, \sigma_2) = \frac{\partial^2 S_{12}(x_1, x_2)}{\partial x_1 \partial x_2} = a(a+1)\left(\sigma_1\sigma_2\left(1 + \frac{x_1}{\sigma_1} + \frac{x_2}{\sigma_2}\right)^{a+2}\right)^{-1},$$

where (a, σ_1, σ_2) is the unknown three-parameter vector of the model, $S_{12}(x_1, x_2)$ is the corresponding bivariate survival function (see (4.4)), and the loglikelihood

TABLE 2 Accidental deaths in Spain, due to crashes (X_1) or other accidental causes (X_2) in a province (or autonomous city). [Table continues on next page.]

	2000		2004		2008		2012	
Province	X_1	X_2	X_1	X_2	X_1	X_2	X_1	X_2
Albacete	64	44	44	48	30	69	17	94
Alicante/Alacant	219	165	198	209	133	255	65	226
Almería	129	75	114	81	56	96	31	103
Araba/Alava	43	27	37	39	19	43	16	53
Asturias	188	181	151	245	74	241	66	297
Avila	19	17	27	37	11	32	15	45
Badajoz	107	57	100	66	64	54	40	99
Balears, Illes	147	124	118	118	93	137	67	164
Barcelona	646	902	414	940	253	1192	226	1101
Bizkaia	168	148	106	168	61	195	51	200
Burgos	92	42	42	87	44	79	31	85
Cáceres	54	45	80	54	33	32	16	58
Cádiz	123	87	129	131	61	133	38	140
Cantabria	69	83	43	101	31	120	24	178
Castellón/Castelló	100	68	87	75	41	74	31	89
Ciudad Real	88	47	60	63	56	73	31	114
Córdoba	91	84	82	98	57	112	47	119
Coruña, A	251	160	169	211	105	229	65	163
Cuenca	33	26	36	32	18	49	18	65
Gipuzkoa	110	109	67	113	49	120	22	136
Girona	94	106	71	119	51	130	48	132
Granada	119	115	112	125	80	134	49	122
Guadalajara	25	18	28	29	15	31	12	44
Huelva	46	42	51	54	44	44	25	62
Huesca	47	27	43	50	19	44	19	49
Jaén	84	72	68	98	75	66	20	80
León	121	72	78	96	69	99	48	121
Lleida	100	61	86	76	56	94	43	75
Lugo	100	73	79	76	50	78	37	85
Madrid	488	665	357	782	253	767	86	696
Málaga	150	135	142	170	113	189	63	240
Murcia	222	132	218	174	118	178	80	172
Navarra	108	68	94	106	52	115	40	114
Ourense	76	69	64	117	40	90	21	94
Palencia	34	19	24	37	17	38	6	44
Palmas, Las	132	142	63	192	111	204	31	101
Pontevedra	186	133	136	211	94	170	70	157

TABLE 2 Continued.

Province/city	2000 X_1	2000 X_2	2004 X_1	2004 X_2	2008 X_1	2008 X_2	2012 X_1	2012 X_2
Rioja, La	60	42	55	62	36	53	14	49
Salamanca	44	34	61	58	36	64	19	51
Santa Cruz de Tenerife	89	109	73	147	59	164	48	170
Segovia	32	15	21	17	11	26	10	18
Sevilla	204	180	216	213	121	213	72	196
Soria	24	13	32	17	12	34	3	17
Tarragona	143	116	130	120	72	157	48	137
Teruel	33	22	23	38	16	40	10	21
Toledo	81	55	79	77	66	74	32	147
Valencia/València	311	274	286	300	183	307	101	360
Valladolid	70	57	66	87	44	71	23	72
Zamora	37	31	28	35	17	31	7	35
Zaragoza	150	86	145	114	85	150	54	117
Ceuta	4	4	9	4	2	6	2	9
Melilla	4	5	3	9	2	8	0	7

Source: Instituto Nacional de Estadistica (2014).

function is given by

$$\begin{aligned}\log \ell(a, \sigma_1, \sigma_2) &= \sum_{i=1}^{n} \log f(x_{1i}, x_{2i}; a, \sigma_1, \sigma_2) \\ &= n \log[a(a+1)] - n \log(\sigma_1) - n \log(\sigma_2) \\ &\qquad - (a+2) \sum_{i=1}^{n} \log\left(1 + \frac{x_{1i}}{\sigma_1} + \frac{x_{2i}}{\sigma_2}\right),\end{aligned}$$

where (x_{1i}, x_{2i}), $i = 1, \dots, n$, is the sample bivariate data, and the maximum likelihood estimation of the parameter vector $(\hat{a}, \hat{\sigma_1}, \hat{\sigma_2})$ is that which maximizes the loglikelihood function $\log \ell(a, \sigma_1, \sigma_2)$.

Finally, we obtained the risk measures for the bivariate Pareto distribution based on Gini's principle, on the proportional hazard transform and on the dual power transform described in Section 4.1, by using (4.6)–(4.8), respectively.

6.2 Results

Table 3 shows the parameter estimates from the bivariate Pareto model (a, σ_1 and σ_2 parameters) fitted to the number of fatalities due to crashes and number of deaths

FIGURE 2 Display of three-dimensional histograms: fatalities due to crashes (X_1) and deaths due to other accidental causes (X_2) in a Spanish province in a year.

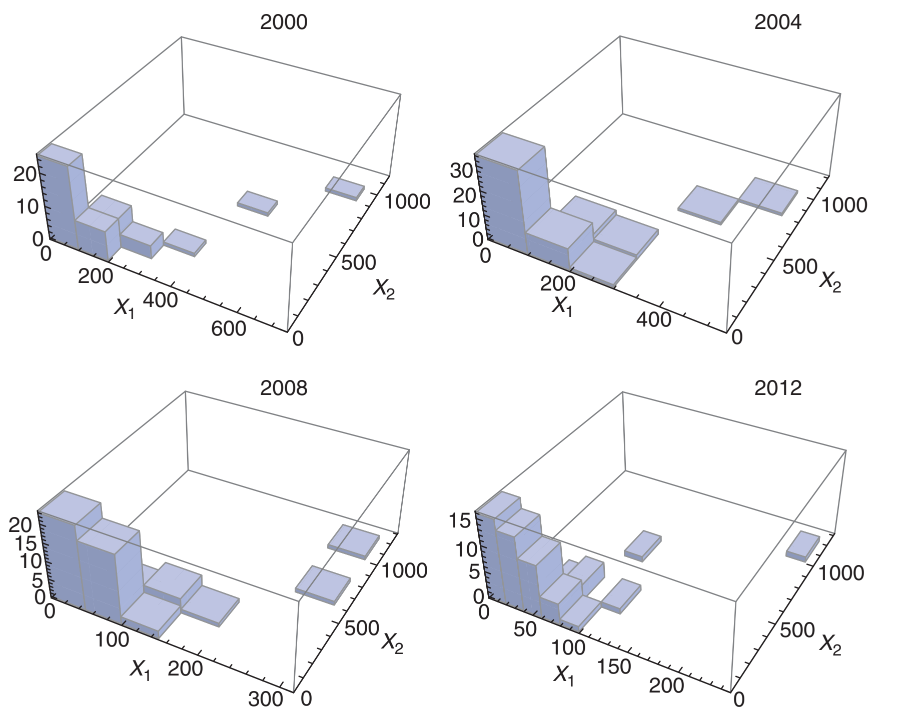


TABLE 3 Parameter estimates from the bivariate Pareto model for the accidental deaths data set by maximum likelihood.

	2000	2004	2008	2012
$\hat{a}$	4.2406	5.7511	5.1390	6.1394
$\hat{\sigma}_1$	395.43	471.63	271.84	206.63
$\hat{\sigma}_2$	321.96	590.56	544.90	699.24

due to other accidental causes in a Spanish province or autonomous city per year, by maximum likelihood, in the four years selected: 2000, 2004, 2008 and 2012.

Tables 4–6 show the risk measures for the bivariate Pareto distribution based on Gini's principle, on the proportional hazard transform and on the dual power transform, respectively.

TABLE 4 Risk measures for the bivariate Pareto distribution based on Gini's principle for accidental death bivariate data.

θ	2000	2004	2008	2012
0.0	17 534.7	15 628.0	11 400.8	6 791.6
0.1	19 025.6	16 911.7	12 348.0	7 346.2
0.2	20 516.5	18 195.4	13 295.2	7 900.7
0.3	22 007.4	19 479.1	14 242.4	8 455.3
0.4	23 498.3	20 762.8	15 189.7	9 009.8
0.5	24 989.2	22 046.5	16 136.9	9 564.3
0.6	26 480.1	23 330.2	17 084.1	10 118.9
0.7	27 971.0	24 613.9	18 031.3	10 673.4
0.8	29 461.8	25 897.6	18 978.5	11 227.9
0.9	30 952.7	27 181.3	19 925.8	11 782.5
1.0	32 443.6	28 465.0	20 873.0	12 337.0

TABLE 5 Risk measures for the bivariate Pareto distribution based on the proportional hazard transform for accidental death bivariate data.

m	2000	2004	2008	2012
1.0	17 534.7	15 628.0	11 400.8	6 791.6
1.1	21 853.0	18 549.4	13 725.9	8 005.9
1.2	27 299.4	21 914.5	16 474.9	9 387.1
1.3	34 308.1	25 814.7	19 755.2	10 966.0
1.4	43 558.0	30 366.8	23 711.6	12 780.9
1.5	56 170.7	35 722.1	28 544.1	14 880.2
1.6	74 136.6	42 080.6	34 536.5	17 325.6
1.7	101 351.0	49 711.5	42 104.9	20 197.3
1.8	146 589.0	58 985.5	51 883.7	23 601.3
1.9	234 589.0	70 427.1	64 889.3	27 680.2
2.0	472 423.0	84 802.8	82 855.8	32 630.6

It can be seen that increasing the value of θ (Table 4) or the value of m (Tables 5 and 6) results in an increase in the corresponding risk measure value. In addition, in this example, it can be seen that risk measures decrease in most cases year-over-year when θ or m is held constant.

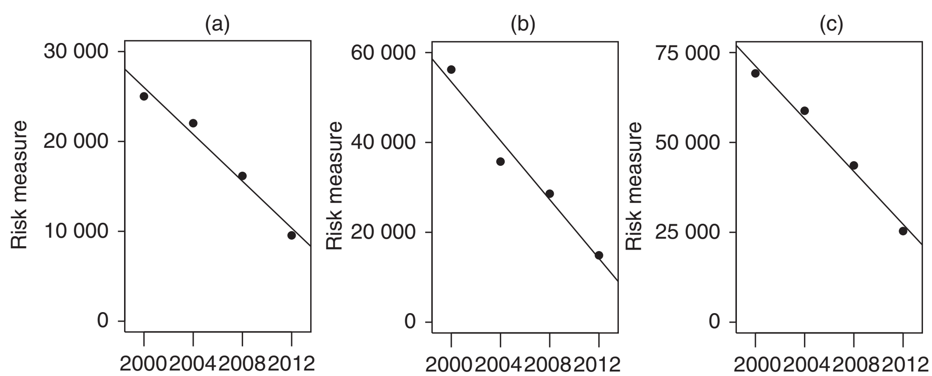
The conclusion for this illustration is that there is evidence of a decrease in the risk for the number of deaths from two different causes from 2000 to 2012.

This application shows that our proposed method to quantify multivariate operational risk is a straightforward method that is useful to monitor multivariate risks.

TABLE 6 Risk measures for the bivariate Pareto distribution based on the dual power transform for accidental death bivariate data.

m	2000	2004	2008	2012
1	17 534.7	15 628.0	11 400.8	6 791.6
2	32 443.6	28 465.0	20 873.0	12 337.0
3	45 739.8	39 634.5	29 182.3	17 141.3
4	57 903.2	49 657.3	36 686.4	21 437.9
5	69 208.8	58 826.7	43 587.7	25 357.8
6	79 832.9	67 327.8	50 014.7	28 983.4
7	89 896.8	75 286.5	56 055.1	32 370.7
8	99 488.7	82 793.2	61 772.2	35 559.8
9	108 675.0	89 915.4	67 213.4	38 580.6
10	117 508.0	96 705.4	72 415.4	41 456.1

FIGURE 3 Trend in risk values based on (a) $\theta = 0.5$ for Gini's principle, (b) $m = 1.5$ for the proportional hazard transform and (c) $m = 5$ for the dual power transform.



When looking at the plots of the distortion functions presented in Figure 1, we clearly see that, for all of them, the larger the parameter, the closer the distortion function is to 1 for low values of t. In the distortion, low values of t correspond exactly to large values of the loss variables. Therefore, we expect to obtain risk measures that increase when the distortion parameter increases. In Tables 4–6, we see that the larger the value of θ and m, the larger the resulting risk value. This happens for all the years (columns) and all the risk measures. The reason is that the weight of the right tail of the loss distribution in the computation of the risk summary value increases with θ and m.

When we look at the risk values by row, we always obtain a decreasing trend. This would not occur if we were using a concave transform, as it would not weight the large value of losses so much.

In Figure 3, we plot the trend in risk values based on $\theta = 0.5$ for Gini's principle, $m = 1.5$ for the proportional hazard transform and $m = 5$ for the dual power transform. These correspond to the middle rows of Tables 4–6, respectively. Such values are powerful indicators, able to capture the multivariate structure of risks and to represent it in a single value per year. When looking at the trend presented in Figure 3, we conclude that there is a clearly decreasing risk over the time period, when the two dimensions of losses are taken into consideration.

We have shown that the multivariate risk measure analysis provides a simple tool to monitor the evolution of risk when we take into account the two dimensions considered in this example: the number of victims by event type. We liked this particular example because it is common to have several types of operational risk events needing to be monitored both over time and simultaneously.

7 CONCLUSIONS

We presented a way to address multivariate distortion risk measures and we have given some examples of distortion functions and distributions where the final expression has a closed form.

We believe that this methodological approach, although it is restricted to nonnegative cases, can be useful in many risk management applications.

The main advantage of our method is that there is no need to use vector-valued risk measures; instead, for some distributions that are typical in the operational risk context, such as the bivariate Pareto, we can obtain analytical expressions for multivariate distortion risk measures. The main drawback of our method is the difficulty in interpreting the summarizing measure in the scale and units of the original components of the vector of losses.

The main limitation regarding interpretation, as in many other aggregation methods, is that distortion functions combine and rescale the original units of measurement. In the multivariate case, when we use distorted multivariate survival functions to obtain a distortion risk measure for a multivariate risk, the units of measurement are also distorted.

DECLARATION OF INTEREST

The authors report no conflicts of interest. The authors alone are responsible for the content and writing of the paper.

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